## Eigenvectors, Eigenvalues, Eigenspaces

**(1)** Let $k$ be a field and let $V, W$ be two vectorspaces over $k$. In MAT240, you proved that the set of all linear transformations from $V$ to $W$, denoted $L(V, W)$ (or sometimes $\mathrm{Hom}_k(V, W)$), equipped with the operations

$$(T_1 + T_2)(v) := T_1(v) + T_2(v);$$
$$(cT)(v) := cT(v)$$

is a vectorspace. Prove this on your own, if you hadn't seen this before. In this tutorial, we will deal with linear transformations whose domain and codomain are the same vectorspace. Let $V$ be a vectorspace and consider the set $L(V, V)$, sometimes denoted $\mathrm{End}(V)$ where End stands for *endomorphism*. Elements of $\mathrm{End}(V)$ are sometimes called linear operators on $V$. You should be able to prove that if $V$ is a finite dimensional vector space of dimension $n$, then $\mathrm{End}(V) \cong M_n(k)$.

**(2)** Let $V$ be a finite dimensional vectorspace and let $\alpha, \beta$ be two bases of $V$. Let $T \in \mathrm{End}(V)$, in other words let $T : V \to V$ be a linear transformation, and let $A, B$ be matrices of this transformation with respect to $\alpha, \beta$, respectively. Show that $\det(A) = \det(B)$. Conclude that we can define the determinant of $T$ to be the determinant of its matrix with respect to any basis. Indeed, no matter what basis we choose, the determinant of the corresponding matrix will be the same. We talked about this in the first tutorial.

**(3)** Let $V$ be a finite dimensional vectorspace and let $T \in \mathrm{End}(V)$. Suppose that there exists a nonzero vector $v \in V$ and a scalar $\lambda \in k$ such that $T(v) = \lambda v$. Show that $v \in \ker(T - \lambda I)$ where $\lambda I : V \to V$ is the linear transformation defined by $(\lambda I)(v) = \lambda v$. Does the converse also hold? Prove that if $\lambda \in k$ is a scalar and if $v \in \ker(T - \lambda I)$, then $T(v) = \lambda v$.

We call a nonzero vector $v \in V$ an **eigenvector** of $T$ associated to the **eigenvalue** $\lambda$ if the above condition holds. That is, if $T(v) = \lambda v$. Or equivalently, if $v \in \ker(T - \lambda I)$.

**(4)** Recall that if $T : V \to W$ is a linear transformation between two vector spaces, then $\ker(T)$ is a subspace of $V$. Prove this on your own, if you hadn't done it earlier. Deduce that $\ker(T - \lambda I)$ in **(3)** is a subspace of $V$. It is called the **eigenspace** associated to the eigenvalue $\lambda$. I will denote it by $E_\lambda$.

**(5)** Show that $\det(T - \lambda I)$ is a polynomial (in $\lambda$) of degree $n$ if $\dim V = n$, (*Hint:* Write the matrix $A$ of $T$ with respect to a fixed basis and observe that $A - \lambda I$ is an $n \times n$ matrix).

**(6)** Let $V = \mathbb{C}^2$ and $T \in \mathrm{End}(\mathbb{C}^2)$ be defined by $T(x, y) = (x, 0)$. Show that $T$ has two eigenvalues: $1, 0$. Show that $(1, 0)$ is an eigenvector associated to 1 and $(0, 1)$ is an eigenvector associated to 0. (You can either do this by observation, or by writing the matrix $A$ of $T$ corresponding to some basis.)

**(7)** Let $T \in \mathrm{End}(V)$. We call a subset $X \subseteq V$ $T$-invariant if $T(V) \subseteq V$. That is, for any $v \in X$, $T(v) \in X$. Show that $v$ is an eigenvector of $T$ if and only if $\mathrm{span}(\{v\})$ is a (1-dimensional) $T$-invariant subspace. Prove that $E_\lambda$ is $T$-invariant if $\lambda$ is an eigenvalue of $T$.

**(8)** Show that if 0 is an eigenvalue of $T$, then $T$ is not one-to-one. Does the converse hold? If yes, prove.

**(9)** (*We didn't have time for this.*) Let $T \in \mathrm{End}(V)$ and $\lambda$ be an eigenvalue of $T$. Show that $\lambda^2$ is an eigenvalue of $T^2$ where by $T^2$ we mean $T \circ T$. Then, generalize this and prove that $\lambda^n$ is an eigenvalue of $T^n$. Show that $\lambda^2 + \lambda + 1$ is an eigenvalue of $T^2 + T + I$. In general, prove that if $p(x)$ is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(T)$.

**(10)** Let $V = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(v) = Av$ where $A$ is the $2 \times 2$ matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Does $T$ have any eigenvalues? Explain this result geometrically.

**(11)** Let $V = M_2(\mathbb{R})$ and $t : V \to V$ be the trace map $A \mapsto A^t$. Find eigenvalues of $t$ and corresponding eigenvectors. (This was a homework problem.)

**(12)** (*We didn't have time for this.*) Suppose that $V$ is an $n$-dimensional vector space. Consider a linear operator $T \in \mathrm{End}(V)$ such that it has $n$-linearly independent eigenvectors: $\{v_1, \ldots, v_n\}$. Since $\dim V = n$, this is a basis of $V$. Write the matrix of this linear transformation with respect to this basis. What do you observe?

**(13)** (*We didn't have time for this.*) Let $T$ be a linear operator on a vector space $V$. Suppose that $\lambda_1, \lambda_2$ are two eigenvalues of $T$, with corresponding eigenvectors $v_1, v_2$. Show that if $\lambda_1 \neq \lambda_2$, then $v_1$ and $v_2$ are linearly independent. The converse of this is not true, identity transformation is a counterexample. We can generalize this result. Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $n$ distinct eigenvalues of a linear operator $T$ on an $n$-dimensional vector space $V$, then $V$ has a basis consisting of eigenvectors of $T$.

**(14)** We call a matrix $A$ *idempotent* if $A^2 = A$. Similarly, we call a linear transformation *idempotent* if $T^2 = T$. Obviously, idempotent matrices are square matrices (Why?). In a MAT240 tutorial last semester, we argued that if $A$ is an idempotent matrix, then either $A = I$ or $A$ is not invertible. Similarly, if $T$ is an idempotent map, then either $T$ is the identity map, or it is not invertible. Today, we will investigate this further in a special case. We will classify the idempotent elements of $\mathrm{End}(\mathbb{R}^2)$, (or of $M_2(\mathbb{R})$ if you wish). So, let $T$ be an idempotent operator which is not the identity.

- **(a)** Show that $\ker(T)$ is non-trivial.
- **(b)** Show that $\dim(E_0)$ is either 1 or 2.
- **(c)** If $\dim(E_0) = 2$, then $T$ is the zero transformation. So, from now on suppose $\dim(E_1) = 1$. This means that there is $v \in \mathbb{R}^2$ such that $T(v) = w \neq 0$. Show that $T(w) = w$.
- **(d)** Deduce that 1 is the other eigenvalue of $T$ and $E_1 = \mathrm{span}(w)$.
- **(e)** Observe that geometrically this means that under the tranformation $T$, the line spanned by the vector $w$ is fixed.
- **(f)** Observe that since $\dim(E_0) = 1$, there is a $u \in \mathbb{R}^2$ such that $E_0 = \mathrm{span}(u)$.
- **(g)** Show that $\{w, u\}$ is a basis of $\mathbb{R}^2$. (See (12) above). This means that any vector $x \in \mathbb{R}^2$ can be written as $au + bw$ for some scalars $a, b \in \mathbb{R}^2$.
- **(h)** Let $x = au + bw \in \mathbb{R}^2$. Compute $T(x)$.
- **(i)** Explain geometrically what (h) means.
- **(j)** You successfully classified all idempotent tranformations.